

# Scalar Quantum Field Theory in Disordered Media

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A free massive scalar field in inhomogeneous random media is investigated. The coefficients of the Klein-Gordon equation are taken to be random functions of the spatial coordinates. The case of an annealed-like disordered medium, modeled by centered stationary and Gaussian processes, is analyzed. After performing the averages over the random functions, we obtain the two-point causal Green's function of the model up to one-loop. The disordered scalar quantum field theory becomes qualitatively similar to a  $\lambda\varphi^4$  self-interacting theory with a frequency-dependent coupling.

## I. INTRODUCTION

Based on results obtained by Ford and collaborators [1, 2] and Hu and Shiokawa [3], recently three of us [4] proposed an analog model for fluctuations of light cones induced by quantum gravity effects. The model is based on two general features of waves propagating in random fluids. First, acoustic perturbations in a fluid define discontinuity surfaces that provide a causal structure with sound cones. Second, propagation of acoustic excitations in a random medium are generally described by wave equations with a random speed of sound [5–8]. Taken together, these features lead to fluctuations of sound cones, analogous to the fluctuations of light cones. Phonons propagating in such a random fluid are then modeling photon propagation in a gravitational field with a fluctuating metric.

In the present paper we depart from the quest for analog models of quantum gravity effects and study quantum fields in a disordered environment in a more general context. Our study belongs to a wide program devoted to propagation of quantum matter fields in a classical background spacetime [9] but with metric fluctuations. Specifically, we consider a scalar quantum field described by a Klein-Gordon-like equation, in which the parameters of the equation, the mass and the coefficient of the second-order time derivative, become random functions of the spatial coordinates. The randomness of the parameters is due to *static* noise sources that couple to the scalar field. In the limit of a vanishing mass, one recovers the random wave equation considered in Ref. [4]. While not addressing the issue of the origin of the noise sources, we mention that one has in mind that they can be induced by a variety of phenomena, like metric fluctuations due to quantum creation of gravitons in a squeezed coherent state in the presence of a black hole or interactions with background topological defects, among others.

Like in the case of quantum fields in the presence of an external heat bath, the noise sources break Lorentz symmetry since they define a preferred reference frame.

A scalar quantum field associated with acoustic waves in a disordered medium can define a situation where sound cones fluctuate randomly. In the study of such a situation, it is important to observe that systems with disorder can be divided into two wide groups, namely, systems with quenched or annealed disorder [10–12]. While in annealed systems the random field is in thermal equilibrium with the others degrees of freedom of the system, in quenched systems they are not. The differentiation between the two types of disorder is an important issue in the studies of the influence of impurities on phase transition phenomena. Many authors have used field theory and the renormalization group to investigate systems with quenched disorders [13], with particular interest in the role played by the disorder on critical exponents. In the case of quenched random fields, by means of the replica trick it is possible to define a quenched generating functional of connected  $n$ -point functions. With this generating functional in hand, the issue of the influence of the disorder on phase transitions can be studied. In the present paper we consider annealed disorder. We consider weak noise fields and implement a perturbative expansion controlled by small parameters that characterize the noise correlation functions. We obtain causal two- and four-point Green's functions of the scalar field. Averaging these Green's functions over the noise fluctuations, one obtains two- and four-point functions qualitatively similar to a self-interacting  $\lambda\varphi^4$  theory. As will be shown, we obtain a frequency-dependent coupling constant.

Since Unruh's original paper [14], the possibility of simulating aspects of general relativity and quantum fields in curved space-time through analog models has been widely discussed in the literature [15, 16]. One interesting proposal is the generation of an acoustic metric in Bose-Einstein condensates and superfluids [17]. On the other hand, analog models with sonic black-hole could find a sort of generalization within the random fluid scenario. Since Bose-Einstein condensates are natural candidates to produce an acoustic black-hole, one way to go beyond the semi-classical approximation is to inquire how these systems behave in the presence of disorder.

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The calculations presented in the present paper are the first steps in the implementation of such a program.

The organization of the paper is as follows: In Section II we discuss the perturbative approach in a free scalar field in the presence of annealed disorder described by two random functions. Section III contains our conclusions. In the appendix some lengthy calculations are presented.

## II. PERTURBATION THEORY IN ANNEALED-LIKE DISORDERED MEDIA

Let us consider a scalar field  $\varphi(t, \mathbf{r})$ , defined in a  $(d+1)$  dimensional space-time, that satisfies the random Klein-Gordon equation:

$$\left[ (1 + \mu) \frac{1}{u_0^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + (1 + \xi) m_0^2 \right] \varphi(t, \mathbf{r}) = 0. \quad (1)$$

The quantities  $\mu = \mu(\mathbf{r})$  and  $\xi = \xi(\mathbf{r})$  are dimensionless random variables, functions of the spatial coordinates only, whose statistical properties will be defined shortly. Here, the random coefficient of the time derivative characterizes the possible discontinuity surfaces of the theory and provides a kind of random causal structure to it. It is well known that the principal part of a differential equation, i.e. the terms with the higher-order derivatives, determines entirely the *loci* of points in space-time where a solution may possess non-null discontinuities. This property is at the root of all analogue models of classical gravitation, namely, an effective causality can be obtained from the kinematical properties of a physical system. The region of influence of excitations is given by an envelope which characterizes the maximum speed of propagation, that is, the light cone of the theory. From a physical point of view the region of influence of the theory may be obtained by an *eikonal* approximation, where the mass term may be disregarded. On the other hand, when the wave frequency is of the same magnitude of the mass term, excitations of the system propagate inside the characteristic cone.

In the present paper we consider zero-mean random functions  $\mu(\mathbf{r})$  and  $\xi(\mathbf{r})$ :

$$\langle \mu(\mathbf{r}) \rangle_\mu = 0, \quad \langle \xi(\mathbf{r}) \rangle_\xi = 0, \quad (2)$$

and, for simplicity, we suppose white-noise correlations:

$$\langle \mu(\mathbf{r}) \mu(\mathbf{r}') \rangle_\mu = \sigma_\mu^2 \delta(\mathbf{r} - \mathbf{r}'), \quad (3)$$

$$\langle \xi(\mathbf{r}) \xi(\mathbf{r}') \rangle_\xi = \sigma_\xi^2 \delta(\mathbf{r} - \mathbf{r}'), \quad (4)$$

where  $\langle \cdots \rangle_\mu$  and  $\langle \cdots \rangle_\xi$  denote ensemble average of noise realizations and  $\sigma_\mu$  and  $\sigma_\xi$  characterize the strengths of the noises. We also suppose that the noises are Gaussian distributed (we use the notation  $\mu(\mathbf{r}_i) = \mu_i$ ):

$$\langle \mu_{i_1} \cdots \mu_{i_{2n}} \rangle_\mu = \langle \mu_{i_1} \mu_{i_2} \rangle_\mu \langle \mu_{i_3} \mu_{i_4} \rangle_\mu \cdots \langle \mu_{i_{2n-1}} \mu_{i_{2n}} \rangle_\mu + \text{permutations}, \quad (5)$$

and correlations of an odd number of noises are zero. The reader may be aware, but it is worth remembering that these random functions  $\mu$  and  $\xi$  are statistically independent.

Since Eq. (1) is linear in  $\varphi(t, \mathbf{r})$ , it is useful to resort to Fourier transforms in order to find its solutions. Therefore, we define the Fourier transform of  $\varphi(t, \mathbf{r})$  on the time variable  $t$ :

$$\varphi(t, \mathbf{r}) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \varphi(\omega, \mathbf{r}), \quad (6)$$

and on the space variable  $\mathbf{r}$ :

$$\varphi(\cdot, \mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot \mathbf{r}} \varphi(\cdot, \mathbf{k}). \quad (7)$$

In addition, we define Fourier transforms of the stationary noise functions:

$$\mu(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k} \cdot \mathbf{r}} \mu(\mathbf{k}), \quad (8)$$

and similarly for  $\xi(\mathbf{r})$ . Using these in Eq. (1), one obtains that the Fourier components of the field  $\varphi(\omega, \mathbf{k})$  satisfy an algebraic equation that can be written as

$$\int d\mathbf{k}' [L_0(\mathbf{k}, \mathbf{k}') + L_1(\mathbf{k}, \mathbf{k}')] \varphi(\omega, \mathbf{k}') = 0, \quad (9)$$

where  $L_0$  is a non-stochastic matrix with elements:

$$L_0(\mathbf{k}, \mathbf{k}') = \left( \frac{\omega^2}{u_0^2} - \mathbf{k}^2 - m_0^2 \right) \delta(\mathbf{k} - \mathbf{k}'), \quad (10)$$

and  $L_1(\mathbf{k}, \mathbf{k}')$  is a stochastic matrix, with elements:

$$L_1(\mathbf{k}, \mathbf{k}') = \frac{1}{(2\pi)^d} \left( \mu(\mathbf{k} - \mathbf{k}') \frac{\omega^2}{u_0^2} - \xi(\mathbf{k} - \mathbf{k}') m_0^2 \right). \quad (11)$$

In  $\mathbf{r}$ -space,  $L_0$  and  $L_1$  read:

$$L_0(\mathbf{r}) = \frac{\omega^2}{u_0^2} + \nabla^2 - m_0^2, \quad (12)$$

$$L_1(\mathbf{r}) = \mu(\mathbf{r}) \frac{\omega^2}{u_0^2} - \xi(\mathbf{r}) m_0^2. \quad (13)$$

As in  $\mathbf{k}$ -space, they act as integral convolution operators. Note that while  $L_1$  is diagonal,  $L_0$  is non-diagonal in  $\mathbf{r}$ -space; yet, in  $\mathbf{k}$ -space the situation is the opposite. In terms of these operators, the random Klein-Gordon equation can be written in matrix form:

$$(L_0 + L_1) \varphi(\omega, \cdot) = 0. \quad (14)$$

From this, one can define the full (operator valued) Green's function  $G$ :

$$G = (L_0 + L_1)^{-1}. \quad (15)$$

Now, if one assumes that the noises are “weak”, a natural perturbative expansion for  $G$  in the form of a Dyson formula can be defined:

$$\begin{aligned} G &= G^{(0)} - G^{(0)} L_1 G^{(0)} + G^{(0)} L_1 G^{(0)} L_1 G^{(0)} + \dots \\ &= G^{(0)} - G^{(0)} \Sigma G^{(0)}, \end{aligned} \quad (16)$$

with the self-energy  $\Sigma$  given by

$$\Sigma = L_1 - L_1 G_0 L_1 + \dots, \quad (17)$$

where  $G^{(0)} = L_0^{-1}$  is the unperturbed (operator valued) Green’s function which, in  $\mathbf{k}$ -space, can be written as

$$G^{(0)}(\omega, \mathbf{k}) = \frac{1}{\omega^2 - (\mathbf{k}^2 + m_0^2) + i\epsilon}. \quad (18)$$

(Hereafter we take  $u_0$  equal to unity.) A schematic representation of the expansion in Eq. (16) is shown in Fig. 1.



FIG. 1: Perturbative expansion of  $G$  in terms of the disorder. The wavy lines represent generically the random functions  $\mu$  and  $\xi$ .

We note that  $iG^{(0)} = \Delta_0$ , where  $\Delta_0$  is the noninteracting Feynman propagator, which is the vacuum expectation value of the time ordered product of the quantum field operators  $\varphi$  at space-time points  $x = (t, \mathbf{x})$  and  $x' = (t', \mathbf{x}')$ :

$$\Delta_0(x, x') = \langle 0 | T[\varphi(t, \mathbf{x}) \varphi(t', \mathbf{x}')] | 0 \rangle. \quad (19)$$

Let us write down explicitly a few terms of the perturbative series in  $\mathbf{r}$ -space, up to second order in the random fields:

$$\begin{aligned} G(\omega, \mathbf{r}, \mathbf{r}') &= G^{(0)}(\omega, \mathbf{r}, \mathbf{r}') - \int d\mathbf{r}_1 G^{(0)}(\omega, \mathbf{r}, \mathbf{r}_1) [\mu(\mathbf{r}_1) \omega^2 - \xi(\mathbf{r}_1) m_0^2] G^{(0)}(\omega, \mathbf{r}_1, \mathbf{r}') \\ &\quad + \int d\mathbf{r}_1 \int d\mathbf{r}_2 G^{(0)}(\omega, \mathbf{r}, \mathbf{r}_2) [\mu(\mathbf{r}_2) \omega^2 - \xi(\mathbf{r}_2) m_0^2] G^{(0)}(\omega, \mathbf{r}_2, \mathbf{r}_1) \\ &\quad \times [\mu(\mathbf{r}_1) \omega^2 - \xi(\mathbf{r}_1) m_0^2] G^{(0)}(\omega, \mathbf{r}_1, \mathbf{r}') + \dots \end{aligned} \quad (20)$$

A pictorial representation in  $\mathbf{r}$ -space of this perturbative expression in terms of diagrams can be done. They correspond to diagrams of multiple-scattering of the Klein-Gordon field on random inhomogeneity scatterers located at positions  $\mathbf{r}_1, \mathbf{r}_2, \dots$ . In  $\mathbf{k}$ -space, similar diagrams correspond to multiple interactions of Fourier components of the Klein-Gordon field and of the random inhomogeneities.

Let us now perform the averages over the random processes that appear in the definition of the propagator  $G$  given in Eq. (20). Note that, because of the Gaussian nature of noise correlations, terms with an odd-number of noise fields do not contribute to the two-point function, and the first nonzero correction to the two-point function comes from the averages over the third term in Eq. (20). Therefore, up to second order in the noise fields, we get

$$\begin{aligned} G^{(1)}(\omega, \mathbf{r}, \mathbf{r}') &\equiv \langle G(\omega, \mathbf{r}, \mathbf{r}') \rangle_{\mu\xi} \\ &= G^{(0)}(\omega, \mathbf{r}, \mathbf{r}') + \bar{G}^{(1)}(\omega, \mathbf{r}, \mathbf{r}'), \end{aligned} \quad (21)$$

with the one-loop correction  $\bar{G}^{(1)}(\omega, \mathbf{r}, \mathbf{r}')$ , represented pictorially in Fig. 2 and given in terms of its Fourier transform  $\bar{G}^{(1)}(\omega, \mathbf{k})$  as

$$\bar{G}^{(1)}(\omega, \mathbf{r}, \mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^d} \bar{G}^{(1)}(\omega, \mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}. \quad (22)$$

This quantity can be written in terms of a self-energy

$\Sigma(\omega, \mathbf{k})$  – defined consistently with Eq. (16) – as:

$$\bar{G}^{(1)}(\omega, \mathbf{k}) = -G^{(0)}(\omega, \mathbf{k}) \Sigma(\omega, \mathbf{k}) G^{(0)}(\omega, \mathbf{k}), \quad (23)$$

with

$$\Sigma(\omega, \mathbf{k}) = -(\sigma_\mu^2 \omega^4 + \sigma_\xi^2 m_0^4) \alpha(\omega), \quad (24)$$

where

$$\alpha(\omega) = \int \frac{d\mathbf{k}}{(2\pi)^d} G^{(0)}(\omega, \mathbf{k}). \quad (25)$$

Note that at one-loop order, the self-energy is actually independent of  $\mathbf{k}$ . From Eq. (16), one has  $G^{-1} = G^{(0)-1} + \Sigma$  and, therefore, in the  $\mathbf{k}$ -representation

$$\begin{aligned} [G^{(1)}(\omega, \mathbf{k})]^{-1} &= [G^{(0)}(\omega, \mathbf{k})]^{-1} + \Sigma(\omega, \mathbf{k}) \\ &= \omega^2 - \mathbf{k}^2 - m^2, \end{aligned} \quad (26)$$

where

$$\begin{aligned} m^2 &= m_0^2 - \Sigma(\omega, \mathbf{k}) \\ &= m_0^2 + (\sigma_\mu^2 \omega^4 + \sigma_\xi^2 m_0^4) \alpha(\omega). \end{aligned} \quad (27)$$

From this result, it is clear that the effect of randomness is to turn a free, conventional scalar quantum field theory into a self-interacting theory, with a self-interaction qualitatively similar to  $\lambda\varphi^4$ . The induced  $\lambda\varphi^4$

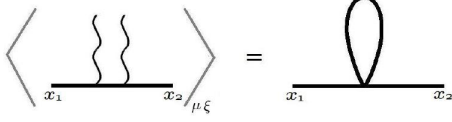


FIG. 2: The one-loop correction to the two-point causal function due to the random inhomogeneities.

theory is a consequence of the Gaussian nature of the functions  $\mu(\vec{r})$  and  $\xi(\vec{r})$ ; a more general polynomial self-interacting model is obtained if non-Gaussian noises are used. The integral is divergent because of the white-noise nature of the noise; a colored noise correlation function would lead to a finite integral.

As mentioned earlier in the text, the induced coupling generated is frequency-dependent,  $\lambda(\omega) \sim \sigma_\mu^2 \omega^4 + \sigma_\xi^2 m_0^4$ . Also, for  $d = 3$  one can isolate the finite part from the integral making use of the identity

$$\frac{|\mathbf{k}|^2}{|\mathbf{k}|^2 - (\omega^2 - m_0^2)} = 1 + \frac{\omega^2 - m_0^2}{|\mathbf{k}|^2 - (\omega^2 - m_0^2)}, \quad (28)$$

so that the finite part of the self-energy,  $\Sigma_f(\omega, \mathbf{k})$ , is given by

$$\begin{aligned} \Sigma_f(\omega, \mathbf{k}) &= (\sigma_\mu^2 \omega^4 + \sigma_\xi^2 m_0^4) \int_0^\infty \frac{d|\mathbf{k}|}{2\pi^2} \frac{\omega^2 - m_0^2}{\omega^2 - |\mathbf{k}|^2 - m_0^2} \\ &= \frac{1}{4\pi} (\sigma_\mu^2 \omega^4 + \sigma_\xi^2 m_0^4) |\omega^2 - m_0^2|^{1/2} \mathcal{A}, \end{aligned} \quad (29)$$

with

$$\mathcal{A} = \begin{cases} +1, & m_0^2 > \omega^2, \\ -i, & \omega^2 > m_0^2. \end{cases} \quad (30)$$

The conclusion from this calculation is that the random fluctuations induce a self-energy with a width, which gives a finite life-time for the excitations.

Since randomness, as mentioned previously, led to an induced coupling, one may ask on the form of the perturbative corrections to the four-point function  $G_4$ . The reason for this particular interest is the following. As well known, given the four-point function, one can define the one-particle-irreducible (1PI) four-point connected Green's function without the external legs  $\Gamma_4$ :

$$\Gamma_4(k_1, k_2, k_3, k_4) = G_4(k_1, k_2, k_3, k_4) \prod_{i=1}^4 \Gamma_{k_i}, \quad (31)$$

where

$$\Gamma_k = \omega^2 - (\mathbf{k}^2 + m_0^2). \quad (32)$$

In a covariant quantum field theory, one may compute the renormalized coupling constant from the proper vertex function, under certain conditions. So, even though we are not dealing with a covariant field theory, it is

natural to expect that such function can bring us some information on the nature of the induced coupling.

Since the random field equation is linear in the field variable, the corresponding action of the system is quadratic in the field, resulting in a Gaussian generating functional. Then, the quantum  $n$ -point Green's functions of the system are products of two-point Green's functions. In particular the four-point Green's function will be of the form:

$$\begin{aligned} G_4(x_1, x_2, x_3, x_4) &= \langle G(x_1, x_2) G(x_3, x_4) \rangle_{\mu\xi} \\ &+ \langle G(x_1, x_3) G(x_2, x_4) \rangle_{\mu\xi} \\ &+ \langle G(x_1, x_4) G(x_2, x_3) \rangle_{\mu\xi}. \end{aligned} \quad (33)$$

As before,  $x = (t, \mathbf{x})$ . Fig. 3 presents a graphical representation of this expression.

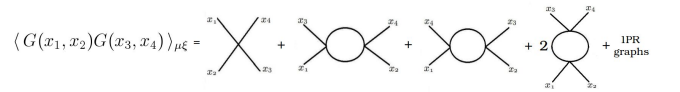


FIG. 3: The four-point Green function up to one-loop, after noise averaging. '1PR graphs' means one-particle reducible graphs.

In Appendix A, Eq. (33) is computed up to one loop; the result can be written as  $G_4(x_1, x_2, x_3, x_4) = G_4^{(0)}(x_1, x_2, x_3, x_4) + G_4^{(1)}(x_1, x_2, x_3, x_4)$ , where  $G_4^{(0)}$  and  $G_4^{(1)}$  are the tree-level and one-loop contributions, respectively. The noise averaging leading to the tree-level four-point function  $G_4^{(0)}$  is illustrated in Fig. 4 and the one-loop corrections  $G_4^{(1)}$  are pictorially represented in Figs. 5 and 6.

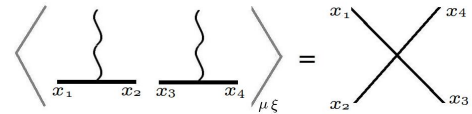


FIG. 4: The disorder-induced, tree-level four-point Green's function.

From Eq. (31) and the results derived in the Appendix, one can express the tree-level proper vertex function as:

$$\begin{aligned} \Gamma_4^{(0)}(k_1, k_2, k_3, k_4) &= (2\pi) \left[ \delta(\omega_3 + \omega_4) (\sigma_\mu^2 \omega_2^2 \omega_4^2 + \sigma_\xi^2 m_0^4) \right. \\ &+ \delta(\omega_2 + \omega_4) (\sigma_\mu^2 \omega_3^2 \omega_4^2 + \sigma_\xi^2 m_0^4) \\ &+ \left. \delta(\omega_2 + \omega_3) (\sigma_\mu^2 \omega_3^2 \omega_4^2 + \sigma_\xi^2 m_0^4) \right]. \end{aligned} \quad (34)$$

As mentioned earlier, the proper understanding of the nature of the induced interacting theory requires the study of the four-point Green's function. We will not dwell into such a study here. Besides being involved (see the Appendix for the one-loop contribution), such a study is out of the scope of the present paper and therefore we reserve it for a future publication.

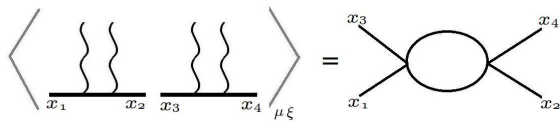


FIG. 5: One-loop corrections of first kind to the four-point Green's function.

The most striking feature that comes out from these calculations is that the induced coupling is qualitatively similar to a  $\lambda\phi^4$  theory, with a frequency-dependent coupling. The important meaning of this is that if one starts with an interacting model, the induced coupling due to the random scatterings in the medium changes the value of the renormalized coupling constant of the model. As a result, a relevant question to be answered is to know what would be the sign of this effective coupling. For instance, if it turns out to be negative, it is clear that we would have a decrease in the value of the renormalized coupling constant. Furthermore, as well known, a negative coupling constant can be a source of instability in field theory, which means that one should look for alternative solutions to circumvent the above mentioned problem. One possibility is to discuss the ground state properties of the system in a theory with metastable vacua. This can be obtained in non-simply connected manifolds. For simplicity one might assume periodic boundary conditions in all  $d + 1$  directions, with compactified lengths  $L_1, L_2, \dots, L_{d+1}$ . At this point we remark that in theories defined on a non-simply connected Euclidean space, one of the compactified length is related to the temperature, remembering the Kubo-Martin-Schwinger (KMS) condition [18]. Therefore in a Euclidean theory describing bosons, we have to impose periodic boundary condition in one compactified direction. In the other  $d$  directions, we are free to choose any boundary conditions. However, imposing periodic boundary conditions in all compactified dimensions enables us to maintain translational invariance [19]. Non-translational invariant systems were studied in Refs. [20]. We take this opportunity to call attention for the results obtained by Arias and co-workers [21], where the thermodynamic of the massless self-interacting scalar field model with negative coupling constant was analyzed. We believe that such results from this reference can be useful to proceed with the investigations of the consequences of impurities in a relativistic model.

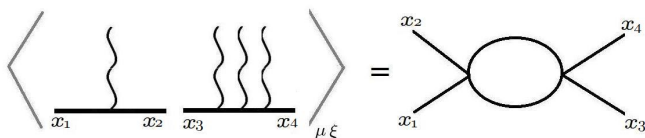


FIG. 6: One-loop corrections of second kind to the four-point Green's function.

### III. CONCLUSIONS

Recently it was proposed that an analog model for quantum gravity effects can be implemented in condensed matter physics when the fluid is a random medium [4]. In order to study the wave propagation of the elementary excitations in the fluid when the acoustic wave propagates in random medium, it was considered that there are no random fluctuation in the density, but only in the reciprocal of the bulk modulus. In this analog model of quantum gravity with a colored noise a free quantum field theory describing phonons becomes a self-interacting model.

In this paper we studied a massive scalar field theory in a disordered medium. We determined the perturbation theory in annealed-like disordered medium, with two distinct random functions. After performing the random averages over the noise function a free scalar quantum field theory became the  $\lambda\phi^4$  self-interacting model. The one-loop two and four-point functions were presented. We obtained that the induced coupling constant is frequency dependent. Note that although we employed Gaussian functions, it is not difficult to extend the method to non-Gaussian functions by means of a cluster expansion [22].

There exist several possibilities to use the above results in analog models. For example, a random fluid with a supersonic acoustic flow can be an analog model that allows one to discuss the effect of the fluctuation of the geometry in the Hawking radiation. Since Bose-Einstein condensate is a natural candidate to produce acoustic black-holes, in order to introduce randomness, i.e., the effects of the metric fluctuations in the acoustic black-hole, we have to investigate how superfluid and Bose-Einstein condensate behave in the presence of disorder. Sound waves in Bose-Einstein condensate, described by random wave equation must reproduce physics beyond the semi-classical approximation in an analog model.

An interesting course of action would be the search of a possible localization of the Hawking thermal flux in such analog model, since it is well known that the effect of the impurities is the localization of classical waves and elementary excitations [23]. In conclusion, the study of quantum fields in disorder medium in the analog models scenario introduces new experimental and theoretical challenges. The study of the relativistic Bose-Einstein condensation in the presence of a random potential [24], and also the consequences of introducing metric fluctuations in the analog model proposed in Ref. [25] is under investigation by the authors.

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### Appendix A: Detailed derivation of the causal two- and four-point functions

We start from the perturbative expansion for the two-point function  $G$ , Eq. (16), which we rewrite as:

$$\begin{aligned} G &= G^{(0)} - G^{(0)} L_1 G^{(0)} + G^{(0)} L_1 G^{(0)} L_1 G^{(0)} + \dots \\ &= G^{(0)} \left( 1 + \sum_{n=1}^{\infty} \mathcal{G}^{(n)} \right), \end{aligned} \quad (\text{A1})$$

with  $\mathcal{G}^{(n)}$  given by

$$\mathcal{G}^{(n)} = (-1)^n \prod_{j=1}^n \left( L_1 G^{(0)} \right)^j, \quad (\text{A2})$$

In terms of time and space variables,  $x = (t, \mathbf{x})$ , this corresponds to

$$G(x, x') = \int dz_1 G^{(0)}(x - z_1) \left[ \delta(z_1 - x') + \sum_{n=1}^{\infty} \mathcal{G}^{(n)}(z_1, x') \right], \quad (\text{A3})$$

and

$$\mathcal{G}^{(n)}(z_1, x') = (-1)^n \prod_{j=1}^n L_1(z_j) \int dz_{j+1} G^{(0)}(z_j, z_{j+1}), \quad (\text{A4})$$

where  $L_1(x)$  is given by

$$L_1(x) = L_1(t, \mathbf{x}) = -\mu(\mathbf{x}) \frac{\partial^2}{\partial t^2} - \xi(\mathbf{x}) m_0^2. \quad (\text{A5})$$

In terms of the frequency and wave-number coordinates in Fourier space,  $(\omega, \mathbf{k})$ ,  $L_1$  is given by Eq. (11). In Eq. (A4), it is to be understood that  $z_{n+1} = x'$  and that there is no integration in  $z_{n+1}$ . In Fig. 7 we present a graphical representation of a generic term in equation (A4).

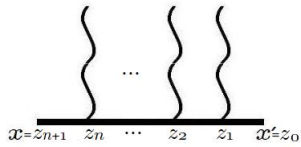


FIG. 7: Graphical representation of a generic term in Eq. (A4)

For completeness, we give here the expression for Fourier transform of the free Green's function  $G^{(0)}(x - x')$ :

$$G^{(0)}(x, x') = \int \frac{dk}{(2\pi)^{d+1}} e^{-ik(x-x')} G^{(0)}(k), \quad (\text{A6})$$

with

$$G^{(0)}(k) = G^{(0)}(\omega, \mathbf{k}) = \Gamma_k^{-1} = \frac{1}{\omega^2 - (\mathbf{k}^2 + m_0^2)}. \quad (\text{A7})$$

From the properties of the noise functions, given in Eqs. (2)-(4), one has that:

$$\langle L_1(x) \rangle_{\mu\xi} = 0, \quad (\text{A8})$$

and

$$\langle L_1(x) L_1(x') \rangle_{\mu\xi} = \left( \sigma_\mu^2 \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial t'^2} + \sigma_\xi^2 m_0^4 \right) \delta(\mathbf{x} - \mathbf{x}'), \quad (\text{A9})$$

From this, it is clear that:

$$\langle \mathcal{G}^{(2n+1)}(x, x') \rangle_{\mu\xi} = 0, \quad (\text{A10})$$

and, therefore, the first correction to the two-point causal Green's function is given by:

$$\bar{G}^{(1)}(x, x') = \int dz_1 G^{(0)}(x, z_1) \langle \mathcal{G}^{(2)}(z_1, x') \rangle_{\mu\xi}. \quad (\text{A11})$$

Now, for  $n = 2$ , Eq. (A4) gives:

$$\begin{aligned} \langle \mathcal{G}^{(2)}(z_1, x') \rangle_{\mu\xi} &= \int dz_2 \langle L_1(z_1) L_1(z_2) \rangle_{\mu\xi} \\ &\times G^{(0)}(z_1, z_2) G^{(0)}(z_2, x'). \end{aligned} \quad (\text{A12})$$

Multiplying this by  $G^{(0)}(x - z_1)$  and integrating over  $z_1$ , one easily obtains the result:

$$\bar{G}^{(1)}(x, x') = \int \frac{dk}{(2\pi)^{d+1}} e^{-ik(x-x')} \bar{G}^{(1)}(k), \quad (\text{A13})$$

with

$$\begin{aligned} \bar{G}^{(1)}(k) &= \bar{G}^{(1)}(\omega, \mathbf{k}) \\ &= -G^{(0)}(\omega, \mathbf{k}) \Sigma(\omega, \mathbf{k}) G^{(0)}(\omega, \mathbf{k}), \end{aligned} \quad (\text{A14})$$

where  $\Sigma(\omega, \mathbf{k})$  is the self-energy given in Eq. (24).

Next, we derive the four-point function. Since the full action of the model is quadratic in the field, the quantum four-point function can be factored into the product of two two-point functions. The average over the noises of this function can then be written as

$$\begin{aligned} G_4(x_1, x_2, x_3, x_4) &= \langle G(x_1, x_2) G(x_3, x_4) \rangle_{\mu\xi} \\ &+ \langle G(x_1, x_3) G(x_2, x_4) \rangle_{\mu\xi} \\ &+ \langle G(x_1, x_4) G(x_2, x_3) \rangle_{\mu\xi}. \end{aligned} \quad (\text{A15})$$

We are interested only in the one-particle irreducible (1PI) parts of this. The lowest-order 1PI contribution is the tree-level four-point function – see Fig. 4. For first term in Eq. (A15) one has:

$$\begin{aligned} \langle G(x_1, x_2) G(x_3, x_4) \rangle_{\mu\xi}^{(0)} &= \int dz_1 \int dz_2 G^{(0)}(x_1, z_1) \\ &\times G^{(0)}(x_3, z_2) \langle \mathcal{G}^{(1)}(z_1, x_2) \mathcal{G}^{(1)}(z_2, x_4) \rangle_{\mu\xi}. \end{aligned} \quad (\text{A16})$$

For  $n = 1$ , Eq. (A4) gives:

$$\begin{aligned} \langle \mathcal{G}^{(1)}(z_1, x_2) \mathcal{G}^{(1)}(z_2, x_4) \rangle_{\mu\xi} &= \langle L_1(z_1) L_1(z_2) \rangle_{\mu\xi} \\ &\times G^{(0)}(z_1, x_2) G^{(0)}(z_2, x_4) \end{aligned} \quad (\text{A17})$$

Using the result of Eq. (A9) and performing the appropriate Fourier transforms, one obtains:

$$\begin{aligned}
\langle G(x_1, x_2)G(x_3, x_4) \rangle_{\mu\xi}^{(0)} &= \int \frac{dk_1}{(2\pi)^{d+1}} \cdots \int \frac{dk_4}{(2\pi)^{d+1}} e^{-i(k_1 x_1 - k_2 x_2 + k_3 x_3 - k_4 x_4)} G^{(0)}(k_1) G^{(0)}(k_2) G^{(0)}(k_3) G^{(0)}(k_4) \\
&\times (\sigma_\mu^2 \omega_2^2 \omega_4^2 + \sigma_\xi^2 m_0^4) \int dz_1 \int dz_2 e^{i(k_1 - k_2)z_1 + i(k_3 - k_4)z_2} \delta(\mathbf{z}_1 - \mathbf{z}_2) \\
&= \int \frac{dk_1}{(2\pi)^{d+1}} \cdots \int \frac{dk_4}{(2\pi)^{d+1}} e^{-i(k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4)} G^{(0)}(k_1) G^{(0)}(k_2) G^{(0)}(k_3) G^{(0)}(k_4) \\
&\times (\sigma_\mu^2 \omega_2^2 \omega_4^2 + \sigma_\xi^2 m_0^4) (2\pi)^{d+1} \delta(k_1 + k_2 + k_3 + k_4) (2\pi) \delta(\omega_3 + \omega_4). \tag{A18}
\end{aligned}$$

To arrive at the last result, we have made the changes  $k_2 \rightarrow -k_2$  and  $k_4 \rightarrow -k_4$ , and used the identity  $\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \delta(\omega_1 + \omega_2) \delta(\omega_3 + \omega_4) = \delta(k_1 + k_2 + k_3 + k_4) \delta(\omega_3 + \omega_4)$ . The other tree-level contributions in Eq. (A15) can be obtained from the result in Eq. (A18) by appropriate changes of integration variables. Specifically, the final result for the tree-level four-point function,  $G_4^{(0)}(x_1, x_2, x_3, x_4)$ , can be written as:

$$\begin{aligned}
G_4^{(0)}(x_1, x_2, x_3, x_4) &= \int \frac{dk_1}{(2\pi)^{d+1}} \cdots \int \frac{dk_4}{(2\pi)^{d+1}} (2\pi)^{d+1} \delta(k_1 + k_2 + k_3 + k_4) \\
&\times e^{-i(k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4)} G_4^{(0)}(k_1, k_2, k_3, k_4), \tag{A19}
\end{aligned}$$

with

$$G_4^{(0)}(k_1, k_2, k_3, k_4) = G^{(0)}(k_1) G^{(0)}(k_2) G^{(0)}(k_3) G^{(0)}(k_4) \Gamma_4^{(0)}(k_1, k_2, k_3, k_4), \tag{A20}$$

where

$$\begin{aligned}
\Gamma_4^{(0)}(k_1, k_2, k_3, k_4) &= (2\pi) \left[ \delta(\omega_3 + \omega_4) (\sigma_\mu^2 \omega_2^2 \omega_4^2 + \sigma_\xi^2 m_0^4) \right. \\
&\quad \left. + \delta(\omega_2 + \omega_4) (\sigma_\mu^2 \omega_3^2 \omega_4^2 + \sigma_\xi^2 m_0^4) + \delta(\omega_2 + \omega_3) (\sigma_\mu^2 \omega_3^2 \omega_4^2 + \sigma_\xi^2 m_0^4) \right]. \tag{A21}
\end{aligned}$$

Next, let us consider the one-loop corrections to the four-point function. Let us focus on contractions which are depicted in Fig. 5. For the first term in Eq. (A15) we have:

$$\langle G(x_1, x_2)G(x_3, x_4) \rangle_{\mu\xi}^{(1-I)} = \int dz_1 \int dz_3 G^{(0)}(x_1, z_1) G^{(0)}(x_3, z_3) \langle \mathcal{G}^{(2)}(z_1, x_2) \mathcal{G}^{(2)}(z_3, x_4) \rangle_{\mu\xi}. \tag{A22}$$

From Eq. (A4), one has:

$$\begin{aligned}
\langle \mathcal{G}^{(2)}(z_1, x_2) \mathcal{G}^{(2)}(z_3, x_4) \rangle_{\mu\xi} &= \int dz_2 \int dz_4 \langle L_1(z_1) L_1(z_2) L_1(z_3) L_1(z_4) \rangle_{\mu\xi} \\
&\times G^{(0)}(z_1, z_2) G^{(0)}(z_2, x_2) G^{(0)}(z_3, z_4) G^{(0)}(z_4, x_4). \tag{A23}
\end{aligned}$$

Using now the fact that the noise correlations are Gaussian :

$$\langle L_1(z_1) L_1(z_2) L_1(z_3) L_1(z_4) \rangle_{\mu\xi} = \langle L_1(z_1) L_1(z_3) \rangle_{\mu\xi} \langle L_1(z_2) L_1(z_4) \rangle_{\mu\xi} + \langle L_1(z_1) L_1(z_4) \rangle_{\mu\xi} \langle L_1(z_2) L_1(z_3) \rangle_{\mu\xi} \tag{A24}$$

The contractions  $\langle L_1(z_1) L_1(z_2) \rangle_{\mu\xi} \langle L_1(z_3) L_1(z_4) \rangle_{\mu\xi}$  do not appear above since they do not contribute to one-loop corrections to the four-point function. Collecting the above results and using Fourier transforms, we have the first contribution to Eq. (A22):

$$\begin{aligned}
\langle G(x_1, x_2)G(x_3, x_4) \rangle_{\mu\xi}^{(1-Ia)} &= \int \frac{dk_1}{(2\pi)^{d+1}} \cdots \int \frac{dk_6}{(2\pi)^{d+1}} e^{-i(k_1 x_1 - k_2 x_2 + k_3 x_3 - k_4 x_4)} G^{(0)}(k_1) G^{(0)}(k_2) G^{(0)}(k_3) G^{(0)}(k_4) \\
&\times G^{(0)}(k_5) G^{(0)}(k_6) (\sigma_\mu^2 \omega_2^2 \omega_4^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega_5^2 \omega_6^2 + \sigma_\xi^2 m_0^4) \\
&\times \int dz_1 \cdots \int dz_4 e^{i(k_1 - k_5)z_1 + i(k_5 - k_2)z_2 + i(k_3 - k_6)z_3 + i(k_6 - k_4)z_4} \delta(\mathbf{z}_1 - \mathbf{z}_3) \delta(\mathbf{z}_2 - \mathbf{z}_4) \\
&= \int \frac{dk_1}{(2\pi)^{d+1}} \cdots \int \frac{dk_6}{(2\pi)^{d+1}} e^{-i(k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4)} G^{(0)}(k_1) G^{(0)}(k_2) G^{(0)}(k_3) G^{(0)}(k_4) \\
&\times G^{(0)}(k_5) G^{(0)}(k_6) (\sigma_\mu^2 \omega_2^2 \omega_4^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega_5^2 \omega_6^2 + \sigma_\xi^2 m_0^4) (2\pi)^{d+1} \delta(k_1 + k_2 + k_3 + k_4) \\
&\times (2\pi)^{d+1} \delta(k_2 + k_4 + k_5 + k_6) (2\pi) \delta(\omega_3 + \omega_4) (2\pi) \delta(\omega_2 + \omega_5). \tag{A25}
\end{aligned}$$

To reach the last result we used the identities:

$$\delta(-\mathbf{k}_1 - \mathbf{k}_3 + \mathbf{k}_5 + \mathbf{k}_6)\delta(\mathbf{k}_2 + \mathbf{k}_4 - \mathbf{k}_5 - \mathbf{k}_6) = \delta(-\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 + \mathbf{k}_4)\delta(\mathbf{k}_2 + \mathbf{k}_4 - \mathbf{k}_5 - \mathbf{k}_6), \quad (\text{A26})$$

$$\begin{aligned} & \delta(\omega_1 - \omega_5)\delta(\omega_5 - \omega_2)\delta(\omega_3 - \omega_6)\delta(\omega_6 - \omega_4) = \delta(\omega_1 - \omega_2)\delta(\omega_3 - \omega_4)\delta(\omega_5 - \omega_2)\delta(\omega_6 - \omega_4) \\ & = \delta(\omega_1 - \omega_2 + \omega_3 - \omega_4)\delta(\omega_3 - \omega_4)\delta(-\omega_2 - \omega_4 + \omega_5 + \omega_6)\delta(\omega_5 - \omega_2), \end{aligned} \quad (\text{A27})$$

and, finally

$$\begin{aligned} & \delta(\omega_1 - \omega_2 + \omega_3 - \omega_4)\delta(\omega_3 - \omega_4)\delta(-\omega_2 - \omega_4 + \omega_5 + \omega_6)\delta(\omega_5 - \omega_2)\delta(-\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 + \mathbf{k}_4)\delta(\mathbf{k}_2 + \mathbf{k}_4 - \mathbf{k}_5 - \mathbf{k}_6) \\ & = \delta(-k_1 + k_2 - k_3 + k_4)\delta(k_2 + k_4 - k_5 - k_6)\delta(\omega_3 - \omega_4)\delta(\omega_5 - \omega_2) \end{aligned} \quad (\text{A28})$$

Then, we have made the changes  $k_2 \rightarrow -k_2$  and  $k_4 \rightarrow -k_4$ .

As for the second contribution to equation (A22), we have:

$$\begin{aligned} \langle G(x_1, x_2)G(x_3, x_4) \rangle_{\mu\xi}^{(1-1b)} &= \int \frac{dk_1}{(2\pi)^{d+1}} \cdots \int \frac{dk_6}{(2\pi)^{d+1}} e^{-i(k_1 x_1 - k_2 x_2 + k_3 x_3 - k_4 x_4)} G^{(0)}(k_1)G^{(0)}(k_2)G^{(0)}(k_3)G^{(0)}(k_4) \\ &\times G^{(0)}(k_5)G^{(0)}(k_6) (\sigma_\mu^2 \omega_2^2 \omega_6^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega_4^2 \omega_5^2 + \sigma_\xi^2 m_0^4) \\ &\times \int dz_1 \cdots \int dz_4 e^{i(k_1 - k_5)z_1 + i(k_5 - k_2)z_2 + i(k_3 - k_6)z_3 + i(k_6 - k_4)z_4} \delta(\mathbf{z}_1 - \mathbf{z}_3)\delta(\mathbf{z}_2 - \mathbf{z}_4) \\ &= \int \frac{dk_1}{(2\pi)^{d+1}} \cdots \int \frac{dk_6}{(2\pi)^{d+1}} e^{-i(k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4)} G^{(0)}(k_1)G^{(0)}(k_2)G^{(0)}(k_3)G^{(0)}(k_4) \\ &\times G^{(0)}(k_5)G^{(0)}(k_6) (\sigma_\mu^2 \omega_2^2 \omega_6^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega_4^2 \omega_5^2 + \sigma_\xi^2 m_0^4) (2\pi)^{d+1} \delta(k_1 + k_2 + k_3 + k_4) \\ &\times (2\pi)^{d+1} \delta(k_2 + k_4 + k_5 + k_6) (2\pi) \delta(\omega_3 + \omega_4) (2\pi) \delta(\omega_2 + \omega_5). \end{aligned} \quad (\text{A29})$$

The last step in the above equation was achieved through the identities:

$$\delta(-\mathbf{k}_1 + \mathbf{k}_4 + \mathbf{k}_5 - \mathbf{k}_6)\delta(\mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_5 + \mathbf{k}_6) = \delta(-\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 + \mathbf{k}_4)\delta(\mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_5 + \mathbf{k}_6), \quad (\text{A30})$$

$$\begin{aligned} & \delta(\omega_1 - \omega_5)\delta(\omega_5 - \omega_2)\delta(\omega_3 - \omega_6)\delta(\omega_6 - \omega_4) = \delta(\omega_1 - \omega_2)\delta(\omega_3 - \omega_4)\delta(\omega_5 - \omega_2)\delta(\omega_3 - \omega_6) \\ & = \delta(\omega_1 - \omega_2 + \omega_3 - \omega_4)\delta(\omega_3 - \omega_4)\delta(-\omega_2 + \omega_3 + \omega_5 - \omega_6)\delta(\omega_5 - \omega_2), \end{aligned} \quad (\text{A31})$$

and, finally

$$\begin{aligned} & \delta(\omega_1 - \omega_2 + \omega_3 - \omega_4)\delta(\omega_3 - \omega_4)\delta(-\omega_2 + \omega_3 + \omega_5 - \omega_6)\delta(\omega_5 - \omega_2)\delta(-\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 + \mathbf{k}_4)\delta(\mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_5 + \mathbf{k}_6) \\ & = \delta(-k_1 + k_2 - k_3 + k_4)\delta(k_2 - k_3 - k_5 + k_6)\delta(\omega_3 - \omega_4)\delta(\omega_5 - \omega_2) \end{aligned} \quad (\text{A32})$$

Similarly as before, we have also made the changes  $k_2 \rightarrow -k_2$ ,  $k_4 \rightarrow -k_4$  and  $k_6 \rightarrow -k_6$ .

The second kind of contractions that also lead to one-loop corrections to the four-point function is shown in Fig. 6. For the first term in Eq. (A15), we have:

$$\langle G(x_1, x_2)G(x_3, x_4) \rangle_{\mu\xi}^{(1-II)} = \int dz_1 \int dz_2 G^{(0)}(x_1, z_1)G^{(0)}(x_3, z_2) \langle \mathcal{G}^{(1)}(z_1, x_2)\mathcal{G}^{(3)}(z_2, x_4) \rangle_{\mu\xi}. \quad (\text{A33})$$

From Eq. (A4), one has:

$$\begin{aligned} \langle \mathcal{G}^{(1)}(z_1, x_2)\mathcal{G}^{(3)}(z_2, x_4) \rangle_{\mu\xi} &= \int dz_2 \int dz_4 \langle L_1(z_1)L_1(z_2)L_1(z_3)L_1(z_4) \rangle_{\mu\xi} \\ &\times G^{(0)}(z_1, x_2)G^{(0)}(z_2, z_3)G^{(0)}(z_3, z_4)G^{(0)}(z_4, x_4). \end{aligned} \quad (\text{A34})$$

Using again the fact that the noise correlations are Gaussian and since we are looking for one-loop corrections, we get:

$$\langle L_1(z_1)L_1(z_2)L_1(z_3)L_1(z_4) \rangle_{\mu\xi} = \langle L_1(z_1)L_1(z_3) \rangle_{\mu\xi} \langle L_1(z_2)L_1(z_4) \rangle_{\mu\xi} \quad (\text{A35})$$



The other contractions of the decomposition above do not appear since they do not contribute to one-loop corrections to the four-point function. Collecting the above results and using Fourier transforms, we have:

$$\begin{aligned}
\langle G(x_1, x_2)G(x_3, x_4) \rangle_{\mu\xi}^{(1-II)} &= \int \frac{dk_1}{(2\pi)^{d+1}} \cdots \frac{dk_4}{(2\pi)^{d+1}} \int \frac{dk}{(2\pi)^{d+1}} \frac{dk'}{(2\pi)^{d+1}} e^{-i(k_1 x_1 - k_2 x_2 + k_3 x_3 - k_4 x_4)} G^{(0)}(k_1) G^{(0)}(k_2) \\
&\times G^{(0)}(k_3) G^{(0)}(k_4) G^{(0)}(k) G^{(0)}(k') (\sigma_\mu^2 \omega_2^2 \omega'^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega^2 \omega_4^2 + \sigma_\xi^2 m_0^4) \\
&\times \int dz_1 \cdots \int dz_4 e^{i(k_1 - k_2)z_1 + i(k_3 - k)z_2 + i(k - k')z_3 + i(k' - k_4)z_4} \delta(\mathbf{z}_1 - \mathbf{z}_3) \delta(\mathbf{z}_2 - \mathbf{z}_4) \\
&= \int \frac{dk_1}{(2\pi)^{d+1}} \cdots \frac{dk_4}{(2\pi)^{d+1}} \int \frac{dk}{(2\pi)^{d+1}} \frac{dk'}{(2\pi)^{d+1}} e^{-i(k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4)} G^{(0)}(k_1) G^{(0)}(k_2) \\
&\times G^{(0)}(k_3) G^{(0)}(k_4) G^{(0)}(k) G^{(0)}(k') (\sigma_\mu^2 \omega_2^2 \omega'^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega^2 \omega_4^2 + \sigma_\xi^2 m_0^4) (2\pi)^{d+1} \\
&\times \delta(k_1 + k_2 + k_3 + k_4) (2\pi)^{d+1} \delta(k_1 + k_2 + k + k') (2\pi) \delta(\omega_1 + \omega_2) (2\pi) \delta(\omega + \omega'). \quad (A36)
\end{aligned}$$

In the last equation we used similar identities of delta functions which were employed in calculation of the expressions (A25) and (A29). Just as the tree-level case, all the one-loop contributions in equation (A15) can be evaluated from the results in equations (A25), (A29) and (A36) by adequate interchanges of variables. Thus, the final result for the one-loop corrections for the four-point function,  $G_4^{(1)}(x_1, x_2, x_3, x_4)$ , reads:

$$\begin{aligned}
G_4^{(1)}(x_1, x_2, x_3, x_4) &= \int \frac{dk_1}{(2\pi)^{d+1}} \cdots \int \frac{dk_4}{(2\pi)^{d+1}} (2\pi)^{d+1} \delta(k_1 + k_2 + k_3 + k_4) \\
&\times e^{-i(k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4)} G_4^{(1)}(k_1, k_2, k_3, k_4), \quad (A37)
\end{aligned}$$

with

$$G_4^{(1)}(k_1, k_2, k_3, k_4) = G^{(0)}(k_1) G^{(0)}(k_2) G^{(0)}(k_3) G^{(0)}(k_4) \Gamma_4^{(1)}(k_1, k_2, k_3, k_4), \quad (A38)$$

where

$$\begin{aligned}
\Gamma_4^{(1)}(k_1, k_2, k_3, k_4) &= (2\pi)^{d+3} \int \frac{dk}{(2\pi)^{d+1}} \int \frac{dk'}{(2\pi)^{d+1}} G^{(0)}(k) G^{(0)}(k') \\
&\times \left[ \delta(k_2 + k_4 + k + k') \delta(\omega_3 + \omega_4) \delta(\omega + \omega_2) (\sigma_\mu^2 \omega_2^2 \omega_4^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega^2 \omega'^2 + \sigma_\xi^2 m_0^4) \right. \\
&+ \delta(k_2 + k_3 + k + k') \delta(\omega_3 + \omega_4) \delta(\omega + \omega_2) (\sigma_\mu^2 \omega^2 \omega_4^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega'^2 \omega_2^2 + \sigma_\xi^2 m_0^4) \\
&+ \delta(k_3 + k_4 + k + k') \delta(\omega_2 + \omega_4) \delta(\omega + \omega_3) (\sigma_\mu^2 \omega_3^2 \omega_4^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega^2 \omega'^2 + \sigma_\xi^2 m_0^4) \\
&+ \delta(k_2 + k_3 + k + k') \delta(\omega_2 + \omega_4) \delta(\omega + \omega_3) (\sigma_\mu^2 \omega^2 \omega_4^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega'^2 \omega_3^2 + \sigma_\xi^2 m_0^4) \\
&+ \delta(k_3 + k_4 + k + k') \delta(\omega_2 + \omega_3) \delta(\omega + \omega_4) (\sigma_\mu^2 \omega_3^2 \omega_4^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega^2 \omega'^2 + \sigma_\xi^2 m_0^4) \\
&+ \delta(k_2 + k_4 + k + k') \delta(\omega_2 + \omega_3) \delta(\omega + \omega_4) (\sigma_\mu^2 \omega^2 \omega_3^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega'^2 \omega_4^2 + \sigma_\xi^2 m_0^4) \\
&+ \delta(k_1 + k_2 + k + k') \delta(\omega_1 + \omega_2) \delta(\omega + \omega') (\sigma_\mu^2 \omega_2^2 \omega'^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega^2 \omega_4^2 + \sigma_\xi^2 m_0^4) \\
&+ \delta(k_1 + k_3 + k + k') \delta(\omega_1 + \omega_3) \delta(\omega + \omega') (\sigma_\mu^2 \omega_3^2 \omega'^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega^2 \omega_4^2 + \sigma_\xi^2 m_0^4) \\
&+ \delta(k_1 + k_4 + k + k') \delta(\omega_1 + \omega_4) \delta(\omega + \omega') (\sigma_\mu^2 \omega_4^2 \omega'^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega^2 \omega_3^2 + \sigma_\xi^2 m_0^4) \\
&+ \delta(k_2 + k_3 + k + k') \delta(\omega_2 + \omega_3) \delta(\omega + \omega') (\sigma_\mu^2 \omega_3^2 \omega'^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega^2 \omega_4^2 + \sigma_\xi^2 m_0^4) \\
&+ \delta(k_2 + k_4 + k + k') \delta(\omega_2 + \omega_4) \delta(\omega + \omega') (\sigma_\mu^2 \omega_4^2 \omega'^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega^2 \omega_3^2 + \sigma_\xi^2 m_0^4) \\
&+ \left. \delta(k_3 + k_4 + k + k') \delta(\omega_3 + \omega_4) \delta(\omega + \omega') (\sigma_\mu^2 \omega_4^2 \omega'^2 + \sigma_\xi^2 m_0^4) (\sigma_\mu^2 \omega^2 \omega_2^2 + \sigma_\xi^2 m_0^4) \right]. \quad (A39)
\end{aligned}$$

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